Discrete-Time Counterparts of Impulsive Hopfield Neural Networks with Leakage Delays

Differential and Difference Equations with Applications pp 351-358 | Cite as

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Conference paper
First Online: 29 July 2013

Part of the Springer Proceedings in Mathematics & Statistics book series (PROMS, volume 47)

Abstract

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   CrossRef (https://doi.org/10.1007/978-94-015-7920-9)
   zbMATH (http://www.emis.de/MATH-item?0752.34039)
   Google Scholar (http://scholar.google.com/scholar_lookup?title=Stability%20and%20Oscillations%20in%20Delay%20Differential%20Equati

Discrete-Time Counterparts of Impulsive Hopfield Neural Networks with Leakage Delays

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Abstract A discrete-time counterpart of a class of Hopfield neural networks with impulses and concentrated and infinite distributed delays as well as a small delay in the leakage terms is introduced. Sufficient conditions for the existence and global exponential stability of a unique equilibrium point of the discrete-time system considered are obtained.

1 Introduction

Hopfield neural networks have found applications in a broad range of disciplines [4–6] and have been studied both in the continuous- and discrete-time cases by many researchers. Moreover, there are many real-world systems and natural processes that behave in a piecewise continuous style interlaced with instantaneous and abrupt changes (impulses). Signal transmission between the neurons causes time delays. Therefore the dynamics of Hopfield neural networks with discrete or distributed delays has a fundamental concern.

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S. Pinelas et al. (eds.), *Differential and Difference Equations with Applications*, Springer Proceedings in Mathematics & Statistics 47, DOI 10.1007/978-1-4614-7333-6_28,

It is known from the literature on population dynamics [1] that time delays in the stabilizing negative feedback terms have a tendency to destabilize the system. Due to some theoretical and technical difficulties [3], so far there have been very few existing works with time delay in leakage (or "forgetting") terms [1, 3, 7, 9].

Our goal in this paper is to introduce a discrete-time counterpart of a class of Hopfield neural networks with impulses and concentrated and infinite distributed delays as well as a small delay in the leakage terms, without essentially changing its stability characteristics. Note that conditions of smallness of the leakage delays have been introduced in [3, 7]. We obtain sufficient conditions for the existence and global exponential stability of a unique equilibrium point of the resulting discrete-time system.

2 Impulsive Continuous-Time Hopfield Neural Network: Existence of a Unique Equilibrium

Consider an impulsive continuous-time neural network consisting of m elementary processing units (or neurons) whose state variables x_i ($i = \overline{1,m}$ which henceforth will stand for $i = 1, 2, \ldots, m$) are governed by the system

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i x_i(t-\sigma) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(t-\tau_{ij}))$$
$$+ \sum_{j=1}^{m} d_{ij} h_j\left(\int_0^{\infty} K_{ij}(s) x_j(t-s)\,\mathrm{d}s\right) + I_i, \quad t > 0, \quad t \neq t_k, \tag{1}$$

$$\Delta x_i(t_k) = B_{ik} x_i(t_k) + \int_{t_k-\sigma}^{t_k} \psi_{ik}(s) x_i(s)\,\mathrm{d}s + \gamma_{ik}, \quad i = \overline{1,m}, \quad k \in \mathbb{N}, \tag{2}$$

with initial values prescribed by piecewise continuous functions $x_i(s) = \phi_i(s)$ which are bounded for $s \in (-\infty, 0]$. In (1), $\sigma > 0$ denotes a delay in the stabilizing (or negative) feedback term $-a_i(x_i - \sigma)$, also called *leakage* or *forgetting term* of the unit i; $f_j(\cdot)$, $g_j(\cdot)$, $h_j(\cdot)$ denote activation functions; the parameters b_{ij}, c_{ij}, d_{ij} are real numbers that represent the weights (or strengths) of the synaptic connections between the jth unit and the ith unit; the real constant I_i represents an input signal introduced from outside the network to the ith unit; τ_{ij} are nonnegative real numbers whose presence indicates the delayed transmission of signals at time $t - \tau_{ij}$ from the jth unit to the unit i; and the delay kernels K_{ij} incorporate the fading past effects (or fading memories) of the jth unit on the ith unit. In (2), $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ denote impulsive state displacements at fixed instants of time t_k ($k \in \mathbb{N}$) involving integral terms whose kernels $\psi_{ik} : [t_k - \sigma, t_k] \to \mathbb{R}$ are measurable functions, essentially bounded on the respective interval. Here it is assumed that $x_i(t_k^+) = \lim_{t \to t_k^+} x_i(t)$ and $x_i(t_k) = x_i(t_k^-) = \lim_{t \to t_k^-} x_i(t)$, and the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$ and $\Delta t_k = t_k - t_{k-1} \geq \theta$, where $\theta > 0$ denotes the minimum time interval between successive impulses. In other words, the value $\theta > 0$ means that the impulses do not occur too often.

The assumptions that accompany the impulsive network (1) and (2) are given as follows:

A$_1$. $0 < a_i < 1/\sigma$, $i = \overline{1,m}$.

A$_2$. The activation functions $f_j, g_j, h_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with respective constants F_j, G_j, H_j, $j = \overline{1,m}$.

A$_3$. $a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| - H_i \sum_{j=1}^{m} |d_{ji}| > 0, \quad i = \overline{1,m}$.

A$_4$. The kernels $K_{ij} : [0,\infty) \to [0,\infty)$ are bounded and piecewise continuous, normalized by $\int_0^{\infty} K_{ij}(s)\,\mathrm{d}s = 1$, and there exists a positive number μ such that $\int_0^{\infty} K_{ij}(s)\,\mathrm{e}^{\mu s}\,\mathrm{d}s < \infty$ for $i, j = \overline{1,m}$.

An equilibrium point of the impulsive network (1) and (2) is denoted by $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ whereby the components x_i^* are governed by the algebraic system

$$a_i x_i^* = \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + \sum_{j=1}^{m} c_{ij} g_j(x_j^*) + \sum_{j=1}^{m} d_{ij} h_j(x_j^*) + I_i, \quad i = \overline{1,m}, \tag{3}$$

and satisfy the linear equations

$$\left(B_{ik} + \int_{t_k - \sigma}^{t_k} \psi_{ik}(s)\,\mathrm{d}s \right) x_i^* + \gamma_{ik} = 0, \quad k \in \mathbb{N},\ i = \overline{1,m}. \tag{4}$$

Lemma 1. *Let $a_i > 0$ $(i = \overline{1,m})$ and conditions* **A**$_2$, **A**$_3$ *be satisfied. Then system (3) has a unique solution $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$.*

In other words, if $a_i > 0$ $(i = \overline{1,m})$ and conditions **A**$_2$–**A**$_4$ are satisfied, the system without impulses (1) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$.

Proof. In system (3) we perform the substitution $y_i = a_i x_i^*$, $i = \overline{1,m}$. Thus, we obtain the system

$$y_i = \Phi_i(y) \equiv \sum_{j=1}^{m} \left[b_{ij} f_j\left(\frac{y_j}{a_j}\right) + c_{ij} g_j\left(\frac{y_j}{a_j}\right) + d_{ij} h_j\left(\frac{y_j}{a_j}\right) \right] + I_i, \ \ i = \overline{1,m}.$$

We can show that the mapping $y \mapsto \Phi(y) = (\Phi_1(y), \Phi_2(y), \ldots, \Phi_m(y))^T$ acts as a contraction in the space $\mathbb{R}^m$ equipped with the norm $\|y\| = \sum_{i=1}^{m} |y_i|$. Thus, it has a unique fixed point y^*. Then $x^* = (y_1^*/a_1, y_2^*/a_2, \ldots, y_m^*/a_m)^T$ is the unique equilibrium point of system (1). □

3 Formulation of a Discrete-Time Impulsive Analogue

Our goal in the present section is to introduce a discrete-time counterpart of system (1) and (2) without essentially changing its stability characteristics. The leakage terms $-a_i x_i(t-\sigma)$ in the right-hand side of (1) make it difficult to apply the semi-discretization procedure described in [2, 8]. Instead, we will discretize all terms in the right-hand side of (1).

Suppose that $\sigma < \theta$. Let the positive integer N be sufficiently large, in particular, such that

$$\left(1+\frac{1}{N}\right) a_i \sigma < 1, \quad i = \overline{1,m}, \qquad \left(1+\frac{1}{N}\right) \sigma < \theta. \tag{5}$$

We choose a discretization step $h = \sigma/N$ and denote by $n = \left[\frac{t}{h}\right]$ the greatest integer in t/h, $\kappa_{ij} = \left[\frac{\tau_{ij}}{h}\right]$ and, for brevity, $x_i(n) = x_i(nh)$, $n \in \mathbb{Z}$. We further replace the integral term $\int_0^\infty K_{ij}(s) x_j(t-s)\,\mathrm{d}s$ $(i, j = \overline{1,m})$ by a sum of the form $\sum\limits_{p=1}^{\infty} \mathcal{K}_{ij}(p) x_j(n-p)$, where the discrete kernels $\mathcal{K}_{ij}(\cdot)$, $i, j = \overline{1,m}$, satisfy the following condition:

A$'_4$. $\mathcal{K}_{ij}(p) \in [0,\infty)$ are bounded for $p \in \mathbb{N}$, normalized by $\sum\limits_{p=1}^{\infty} \mathcal{K}_{ij}(p) = 1$, and there exists a number $\nu > 1$ such that $\sum\limits_{p=1}^{\infty} \mathcal{K}_{ij}(p) \nu^p < \infty$.

Thus, we obtain the following discretization of the right-hand side of (1):

$$-a_i x_i(n-N) + \sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(n-\kappa_{ij}))$$
$$+ \sum_{j=1}^{m} d_{ij} h_j \left(\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) x_j(n-p) \right) + I_i, \quad n \in \mathbb{N}, \quad i = \overline{1,m}.$$

The negative sign of the first term makes difficult the use of Lyapunov's functionals as in [2, 8]. We eliminate this term by using for σ small enough the approximation

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}(nh) \approx \frac{1 - Nha_i}{h} x_i(n+1) - \frac{1-(N+1)ha_i}{h} x_i(n) - a_i x_i(n-N).$$

Let us recall that $Nha_i = \sigma a_i < 1$ by condition **A**$_1$ and $(N+1)ha_i = \left(1+\frac{1}{N}\right)\sigma a_i < 1$ by virtue of (5). Thus, we obtain the following discrete-time analogue of the system without impulses (1):

$$
\begin{aligned}
&(1-Nha_i)x_i(n+1)\\
&\quad=(1-(N+1)ha_i)x_i(n)+h\left(\sum_{j=1}^{m}b_{ij}f_j(x_j(n))\right.\\
&\qquad\left.+\sum_{j=1}^{m}c_{ij}g_j(x_j(n-\kappa_{ij}))+\sum_{j=1}^{m}d_{ij}h_j\left(\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)x_j(n-p)\right)+I_i\right),
\end{aligned}
\tag{6}
$$

$n\in\mathbb{N}$, $i=\overline{1,m}$, with initial values of the form $x_i(-\ell)=\phi_i(-\ell)$ $(\ell\in\{0\}\cup\mathbb{N})$, where the sequences $\{\phi_i(-\ell)\}_{\ell=0}^{\infty}$ are bounded for all $i=\overline{1,m}$.

Next we discretize the impulse conditions (2). If we denote $n_k=\left[\frac{t_k}{h}\right]$, $k\in\mathbb{N}$, we obtain a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ satisfying $0<n_1<n_2<\cdots<n_k\to\infty$ as $k\to\infty$ and $\Delta n_k=n_k-n_{k-1}\geq\left[\frac{\theta}{h}\right]-1$. With each such integer n_k we associate two values of the solution $x(n)$, namely, $x(n_k)$ which can be regarded as the value of the solution before the impulse effect and whose components are evaluated by (6) and $x^+(n_k)$ which can be regarded as the value of the solution after the impulse effect and whose components are evaluated by the equations

$$
x_i^+(n_k)-x_i(n_k)=\sum_{\ell=n_k-N}^{n_k}B_{ik\ell}x_i(\ell)+\gamma_{ik},\qquad i=\overline{1,m},\quad k\in\mathbb{N},\tag{7}
$$

where $B_{ik\ell}$ are suitably chosen constants.

Further on we will call system (6) and (7) the discrete-time analogue of the system with impulses (1) and (2).

The components of an equilibrium point $x^*=(x_1^*,x_2^*,\ldots,x_m^*)^T$ of system (6) and (7) must satisfy the (3) and

$$
\sum_{\ell=n_k-N}^{n_k}B_{ik\ell}x_i^*+\gamma_{ik}=0.\tag{8}
$$

To ensure that systems (1), (2) and (6), (7) have the same equilibrium points if any, we choose the constants $B_{ik\ell}$ so that

$$
\sum_{\ell=n_k-N}^{n_k}B_{ik\ell}=B_{ik}+\int_{t_k-\sigma}^{t_k}\psi_{ik}(s)\,\mathrm{d}s,\qquad i=\overline{1,m},\quad k\in\mathbb{N}.
$$

Definition 1. The equilibrium point $x^*=(x_1^*,x_2^*,\ldots,x_m^*)^T$ of system (6) and (7) is said to be *globally exponentially stable with a multiplier* ρ if there exist constants $M>1$ and $\rho\in(0,1)$, and any other solution $x(n)=(x_1(n),x_2(n),\ldots,x_m(n))^T$ of system (6) and (7) is defined for all $n\in\mathbb{N}$ and satisfies the estimate

$$
\sum_{i=1}^{m}|x_i(n)-x_i^*|\leq M\rho^n\sum_{i=1}^{m}\sup_{\ell\in\{0\}\cup\mathbb{N}}|x_i(-\ell)-x_i^*|.\tag{9}
$$

4 Main Results: Sufficient Conditions for Global Exponential Stability of the Equilibrium Point

Theorem 1. *Let systems (6) and (7) satisfy conditions* $\mathbf{A}_1$–$\mathbf{A}_3$, $\mathbf{A}_4'$, (5)*, and the components of the unique equilibrium point* $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ *of system (6) satisfy (8). Then there exist constants* $M' > 1$ *and* $\lambda \in (1, \nu]$ *such that any other solution* $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T$ *of system (6) and (7) is defined for all* $n \in \mathbb{N}$ *and satisfies the estimate*

$$\sum_{i=1}^{m} |x_i(n) - x_i^*| \le M' \lambda^{-n} \prod_{k=1}^{i(1,n)} B_k' \sum_{i=1}^{m} \sup_{\ell \in \{0\} \cup \mathbb{N}} |x_i(-\ell) - x_i^*|, \tag{10}$$

$$i(1,n) = \begin{cases} 0, & n \le n_1, \\ \max\{k \in \mathbb{N} : n_k < n\}, & n > n_1, \end{cases} \quad B_k' = B_k \left(1 + c \max_{i=\overline{1,m}} (1 - ca_i)^{-1}\right) \text{ and}$$

$$B_k = \max_{i=\overline{1,m}} \max\left\{ \left|1 + B_{ikn_k}\right|, \max_{n_k - N \le \ell \le n_k - 1} |B_{ik\ell}| \right\}, \quad k \in \mathbb{N}.$$

Proof. From the conditions of the theorem it follows that system (6) and (7) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$. For any $n \in \mathbb{N} \cup \{0\}$, from (6) and (3), by virtue of condition $\mathbf{A}_2$, we derive the inequalities

$$\begin{aligned} (1 - Nha_i)|x_i(n+1) - x_i^*| \le (1 - (N+1)ha_i)|x_i(n) - x_i^*| \\ + h \sum_{j=1}^{m} \Bigg\{ |b_{ij}| F_j |x_j(n) - x_j^*| + |c_{ij}| G_j |x_j(n - \kappa_{ij}) - x_j^*| \\ + |d_{ij}| H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) |x_j(n-p) - x_j^*| \Bigg\}, \quad i = \overline{1,m}. \end{aligned}$$

For $\lambda \in [1, \nu]$, let us denote $y_i(n) = \lambda^n |x_i(n) - x_i^*|$, $n \in \mathbb{Z}$, and define a Lyapunov functional $V(\cdot)$ by

$$\begin{aligned} V(n) = \sum_{i=1}^{m} \Bigg\{ (1 - \sigma a_i) y_i(n) + h \sum_{j=1}^{m} \Bigg[|c_{ij}| G_j \lambda^{\kappa_{ij}+1} \sum_{\ell = n - \kappa_{ij}}^{n-1} y_j(\ell) \\ + |d_{ij}| H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) \lambda^{p+1} \sum_{\ell = n-p}^{n-1} y_j(\ell) \Bigg] \Bigg\}. \end{aligned} \tag{11}$$

~~AQ1~~ It is easy to see that $V(n) \ge 0$ for $n \in \mathbb{N} \cup \{0\}$ and $V(0) < \infty$ by $\cancel{\mathbf{A}_6'}$. More precisely,

$$V(0) \le M \sum_{i=1}^{m} \sup_{\ell \in \mathbb{N} \cup \{0\}} |x_i(-\ell) - x_i^*|, \tag{12}$$

where

$$M = \max_{i=\overline{1,m}} \left\{ 1 - \sigma a_i + h \left[G_i \sum_{j=1}^{m} |c_{ji}| \lambda^{\kappa_{ji}+1} + H_i \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathcal{K}_{ji}(p) \lambda^{p+1} \right] \right\}.$$

Further on, we obtain

$$V(n+1) - V(n) \le -\sum_{i=1}^{m} \Psi_i(\lambda) y_i(n),$$

where $\Psi_i(\lambda) = h\lambda \left(a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| \lambda^{\kappa_{ji}} - H_i \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathcal{K}_{ji}(p) \lambda^{p} \right)$ $-(\lambda-1)(1-\sigma a_i)$. By condition $\mathbf{A}_4'$ the functions $\Psi_i(\lambda)$ $(i = \overline{1,m})$ are well defined and continuous for $\lambda \in [1, \nu]$. Moreover, $\Psi_i(1) = h \left(a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| - H_i \sum_{j=1}^{m} |d_{ji}| \right) > 0$, $i = \overline{1,m}$, by virtue of $\mathbf{A}_4'$ and $\mathbf{A}_3$. By continuity, for each $i = \overline{1,m}$, there exists a number $\lambda_i \in (1, \nu]$ such that $\Psi_i(\lambda) \ge 0$ for $\lambda \in (1, \lambda_i]$. If we denote $\lambda_0 = \min_{i=\overline{1,m}} \lambda_i$, then $\lambda_0 > 1$ and $\Psi_i(\lambda) \ge 0$ for $\lambda \in (1, \lambda_0]$ and $i = \overline{1,m}$. This implies $V(n+1) \le V(n)$ for $n \ne n_k$ and $V(n_k+1) \le V^+(n_k)$, where $V^+(n_k)$ contains $|x_i^+(n_k) - x_i^*|$ instead of $|x_i(n_k) - x_i^*|$. The above inequalities yield

$$V(n) \le \begin{cases} V^+(n_k) & \text{for } n_k < n \le n_{k+1}, \\ V(0) & \text{for } 0 < n \le n_1. \end{cases} \tag{13}$$

From equalities (7) and (8), we find

$$|x_i^+(n_k) - x_i^*| \le B_k \sum_{\ell=n_k-N}^{n_k} |x_i(\ell) - x_i^*|,$$

where the constants B_k were introduced in the statement of Theorem 1, and

$$V^+(n_k) \le B_k' V^+(n_{k-1})$$

for $k \ge 2$ and, similarly, $V^+(n_1) \le B_1' V(0)$.

Combining the last inequalities and (13), we derive the estimate

$$V(n) \le \prod_{k=1}^{i(1,n)} B_k' V(0). \tag{14}$$

Finally, from the inequalities

$$\sum_{i=1}^{m} |x_i(n) - x_i^*| \leq \max_{i=\overline{1,m}} (1 - ca_i)^{-1} \lambda^{-n} V(n),$$

(14) and (12) we deduce (10) with $M' = M \max_{i=\overline{1,m}} (1 - ca_i)^{-1}$ and any $\lambda \in (1, \lambda_0]$. □

For three sets of additional assumptions, we show that inequality (10) implies global exponential stability of the equilibrium point x^* of the discrete-time system (6) and (7).

Corollary 1. *Let all conditions of Theorem 1 hold. Suppose that $B_k' \leq 1$ for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point x^* of the discrete-time system (6) and (7) is globally exponentially stable with multiplier $1/\lambda_0$.*

Corollary 2. *Let all conditions of Theorem 1 hold and $\limsup_{n\to\infty} \frac{i(1,n)}{n} = p < +\infty$. Let there exist a positive constant B such that $B_k' \leq B$ for all sufficiently large values of $k \in \mathbb{N}$ and $B^p < \lambda_0$. Then for any $\rho \in \left(\frac{B^p}{\lambda_0}, 1\right)$ the equilibrium point x^* of the discrete-time system (6) and (7) is globally exponentially stable with multiplier ρ.*

Corollary 3. *Let all conditions of Theorem 1 hold. Suppose that there exists a constant $\mu \in (1, \lambda_0)$ such that $B_k' \leq \mu^{n_k - n_{k-1}}$ for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point x^* of the discrete-time system (6) and (7) is globally exponentially stable with multiplier μ/λ_0.*

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